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Linearization of Moffat's Symmetric Complex Metric Gravity

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ABSTRACT: In this paper we investigate a complex symmetric generalization of general relativity and in particular we investigate its linearized field equations. We begin by reviewing some basic definitions and structures in Moffat's symmetric complex metric field theory of gravity. We then move on to derive the linearized retarded complex field equations. In addition to this we also derive a linearization of Moffat's field equations based on the more rigorous Fermi coordinate approach. In conclusion it is shown that the linearized symmetric complex field equations leads to a complex form of gravitomagnetism. We also briefly review the gravitational wave equation from the source less linearized symmetric complex field equations and discuss some open problems.

KEYWORDS: Classical Theories of Gravity.

Contents

1. Introduction

General relativity is arguably the most successful theory of physics in the 20:th century [7]. A multitude of attempts to modify general relativity in order to incorporate other forces and quantum mechanics has therefore been carried out [7]. The complex metric generalization of general relativity has an extensive history [2, 4, 5, 6, 8, 14, 15, 16, 17]. In an attempt to unify gravity and electromagnetism Einstein introduced the complex generalization of the metric tensor in 1945 [5, 6]. Recently this field of complex metric gravity has drawn attention for a number of reasons [2, 4, 14]. One reason is that there are promising leads to a quantum gravity in the complex metric formalism and another is that there are signs within string theory that the metric by necessity needs to be complex [14]. Although most approaches have regarded the investigation of hermitian metric tensors there are some studies that have focused on symmetric ones [8, 14, 15, 16, 17]. The first studies on the possibility of a symmetric complex metric theory as a unified field theory were done by Moffat [15, 16, 17]. The physical interpretation of these symmetric complex metric field theories is still an open problem and in this paper the symmetric complex field is considered a modification of general relativity [14]. In this paper we shall study the linearized field of the symmetric complex metric field theory proposed by Moffat [15, 16].

2. Moffat's symmetric complex metric approach

This symmetric complex metric field theory is based on Moffat's approach [14, 15, 16, 17]. In general the spacetime manifold is complex with coordinates (See section 3 [14]):

$$z^{\mu} = x^{\mu} + iy^{\mu}, \tag{2.1}$$

which can be generalized with hyperbolic complex coordinates in order to avoid ghosts in the action [14]. In that case the field can be formulated as an 8-dimensional real spacetime [14]. In this approach the metric tensor has the form:

$$g'_{\mu\nu} = g_{\mu\nu} + ik\alpha_{\mu\nu}, \tag{2.2}$$

where $g_{\mu\nu}$ and $\alpha_{\mu\nu}$ are real-valued tensors and k is some real-valued constant and generally $'$ denotes complex valued entities. The metric shall be considered symmetric:

$$g_{\mu\nu} = g_{\nu\mu}, \qquad \alpha_{\mu\nu} = \alpha_{\nu\mu}, \qquad g'_{\mu\nu} = g'_{\nu\mu}. \tag{2.3}$$

This gives the line element:

$$ds^2 = g'_{\mu\nu}dx^\mu dx^\nu = (g_{\mu\nu} + ik\alpha_{\mu\nu})dx^\mu dx^\nu. \tag{2.4}$$

The component $g_{\mu\nu}$ is the *gravitational metric tensor* (equivalent to the one defined in general relativity) and $\alpha_{\mu\nu}$ shall be referred to as the *imaginary metric tensor*. The definition of the covariant metric is given by the usual definition (although the metric here is complex):

$$g'_{\mu\nu}g'^{\mu\rho} = \delta^\rho_\nu, \tag{2.5}$$

as well as we shall require (In accordance with the approach by Moffat [14, 15, 16, 17]):

$$g_{\mu\nu}g^{\mu\rho} = \delta^\rho_\nu. \tag{2.6}$$

The definition of the field equations shall remain equivalent to the Einstein field equations (EFE), apart from the fact that the metric makes it complex:

$$R'_{\mu\nu} - \frac{1}{2}g'_{\mu\nu}R' = \frac{8\pi G}{c^4}T'_{\mu\nu}. \tag{2.7}$$

These will be referred to as *Moffat's symmetric complex metric field equations* or simply *the complex field equations*. Here the Ricci tensor, Ricci scalar and the stress-energy tensor are complex. The construction of the complex metric field tensor (2.4) can also be characterized via complex vielbein $E'^a_\mu = Re[E'^a_\mu] + iIm[E'^a_\mu]$:

$$g'_{\mu\nu} = E'^a_\mu E'^b_\nu \eta_{ab}. \tag{2.8}$$

This type of construction has been used (with both real and complex vielbeins) in order to reformulate general relativity as a gauge theory [2, 4]. The determination of Christoffel symbols follows from:

$$g'_{\mu\nu;\lambda} = \partial_\lambda g'_{\mu\nu} - g'_{\rho\nu}\Gamma'^\rho_{\mu\lambda} - g'_{\mu\rho}\Gamma'^\rho_{\nu\lambda} = 0. \tag{2.9}$$

The complex Riemann tensor appears like:

$$R'^\lambda_{\mu\nu\sigma} = -\partial_\sigma\Gamma'^\lambda_{\mu\nu} + \partial_\nu\Gamma'^\lambda_{\mu\sigma} + \Gamma'^\lambda_{\rho\nu}\Gamma'^\rho_{\mu\sigma} - \Gamma'^\lambda_{\rho\sigma}\Gamma'^\rho_{\mu\nu} \tag{2.10}$$

and its contracted curvature tensor (Ricci tensor):

$$R'^\sigma_{\mu\nu\sigma} = R'_{\mu\nu}. \tag{2.11}$$

Furthermore we may conclude that there will be four complex (equivalent to eight real) Bianchi identities [14, 15, 16, 17]:

$$(R'^{\mu\nu} - \frac{1}{2}g'^{\mu\nu}R')_{;\nu} = 0. \tag{2.12}$$

Any vector transported in this complex metric field will have real and imaginary components. In fact, if we look at the covariant derivative of a complex vector A'_μ we get:

$$A'_{\mu;\nu} = A'_{\mu,\nu} - A'_\alpha\Gamma'^\alpha_{\mu\nu}, \tag{2.13}$$

which, if we let $A'_\mu = V_\mu + ikA_\mu$ then (2.13) breaks up into the two equations:

$$V_{\mu;\nu} = V_{\mu,\nu} + V_\alpha Re[\Gamma'^\alpha_{\mu\nu}] - kA_\alpha Im[\Gamma'^\alpha_{\mu\nu}], \tag{2.14}$$

$$A_{\mu;\nu} = A_{\mu,\nu} + A_\alpha Re[\Gamma'^\alpha_{\mu\nu}] + V_\alpha Im[\Gamma'^\alpha_{\mu\nu}]. \tag{2.15}$$

This shows that the transport of any real valued vector can result in a complex valued one. Also it shows the interaction between real and imaginary components of a vector along a transport. Generally Moffat's complex field equations can be derived from a complex Einstein-Hilbert action [16, 17]:

$$S' = \frac{c^4}{16\pi G} \int d^4x(\sqrt{-g'}g'^{\mu\nu}R'_{\mu\nu}). \tag{2.16}$$

It is a general belief that S' has to be real under complex metric circumstances [5, 8, 14, 15, 16, 17]. Indeed, if we apply the variational principle applied to the complex Einstein-Hilbert action with matter term (2.16) one gets the correct complex field equations [15, 16, 17].

3. Correspondence to real and imaginary gravity

3.1 Real gravity

The real gravitational part becomes visible when the imaginary tensor components $\alpha_{\mu\nu}$ vanishes:

$$g'_{\mu\nu} = g_{\mu\nu}. \tag{3.1}$$

This scenario reduces the complex field theory (2.7) to being equivalent to the Einstein field equations of general relativity:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{3.2}$$

which are real valued.

3.2 Imaginary gravity

If we let the gravitational metric tensor $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$, which is the traditional vacuum situation in general relativity, we get a form of imaginary gravity instead:

$$g'_{\mu\nu} = \eta_{\mu\nu} + ik\alpha_{\mu\nu}. \tag{3.3}$$

This situation is not pure vacuum since the complex metric tensor $\alpha_{\mu\nu}$ will still affect the background field. If we in the most extreme case let $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ and $\alpha_{\mu\nu} \rightarrow 0$ then we get a pure form of vacuum state where no gravitational and no imaginary metric effects are present:

$$g'_{\mu\nu} = \eta_{\mu\nu}. \tag{3.4}$$

This field is governed by the Minkowski metric.

4. Linearized symmetric complex field theory

4.1 Linearized retarded field equations

This linearized retarded field equations approach was used by Moffat in his original work on symmetric complex metric gravity [15, 16, 17]. Assume that the metric tensor can be expressed as the Minkowski component and a linear perturbation as:

$$g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}. \tag{4.1}$$

It is then useful to define the trace-reversed amplitude as:

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h', \tag{4.2}$$

where $h' = \eta^{\mu\nu}h'_{\mu\nu}$ is the trace of $h'_{\mu\nu}$. The linearized version of the complex field equations (2.7) appears as follows:

$$\Box\bar{h}'_{\mu\nu} = -\frac{16\pi G}{c^4}T'_{\mu\nu}, \tag{4.3}$$

after imposing the Lorenz gauge condition $\bar{h}'^{\mu\nu}_{,\nu} = 0$ [11]. If in addition to linearizing the field equations assume that $|v| << c$ it is useful to consider the retarded solution as follows:

$$\bar{h}'_{\mu\nu} = \frac{4G}{c^4}\int \frac{T'_{\mu\nu}(ct - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'. \tag{4.4}$$

This lets us define $T'^{00} = \rho' c^2$ and $T^{0i} = cj'^i$, with complex density and complex matter current defined as ρ' and $\vec{j}'$ respectively. It is also useful here to define the complex Gravitomagnetic (GM) potentials $(\Phi', \vec{A}')$ as $\bar{h}'_{00} = 4\Phi'/c^2$, $\bar{h}'_{0i} = -2A'_i/c^2$ and $\bar{h}'_{ij} = O(c^{-4})$. The transverse gauge condition implies:

$$\frac{1}{c}\frac{\partial\Phi'}{\partial t} + \frac{1}{2}\nabla\cdot\vec{A}' = 0. \tag{4.5}$$

Furthermore the wave equations of the $0i$-components become:

$$\Box\Phi' = -4\pi G\rho', \tag{4.6}$$

$$\Box A'_i = \frac{8\pi G}{c}j'^i. \tag{4.7}$$

These equations make up a complex analogue of Maxwell's equations, and are the complex gravitomagnetic field equations. This linearized retarded approach is only valid in certain frames and it does not bring the correct stress-energy tensor for neither standard nor complex linearized gravity [10]. This is the primary reason why we shall address this more generally via a Fermi coordinate approach below.

4.2 Fermi coordinate approach to linear complex symmetric gravity

In this section we are going to use Mashhoon's Fermi approach to gravitomagnetism and apply it to the complex field equations (2.7). In the local reference frame one may setup a Fermi coordinate system along its path. This is equivalent to constructing a an inertial system of coordinates in the immediate neighborhood [10]. One may then let $\lambda^{\mu}_{(\alpha)}$ be the non-rotating orthonormal tetrad of the reference observer (here taken to be real valued). $\lambda^{\mu}_{(\alpha)}$ is a collection of unit vectors along ideal gyro directions that are parallel transported along the worldline [10]. The new space will have Fermi coordinates $X^{\mu} = (T, \vec{X})$. The Riemann tensor will be projected on the orthonormal tetrad of the reference observer according to:

$$R'_{\alpha\beta\gamma\delta} = R'_{\mu\nu\rho\sigma}\lambda^{\mu}_{(\alpha)}\lambda^{\nu}_{(\beta)}\lambda^{\rho}_{(\gamma)}\lambda^{\sigma}_{(\delta)}. \tag{4.8}$$

This gives the space time metric:

$$g'_{00} = -1 - R'_{0i0j}X^iX^j + ..., \tag{4.9}$$

$$g'_{0i} = -\frac{2}{3}R'_{0jik}X^jX^k + ..., \tag{4.10}$$

$$g'_{ij} = \delta_{ij} - \frac{1}{3}R'_{ikjl}X^kX^l + ... \tag{4.11}$$

These coordinates are admissible within a cylindrical spacetime of radius $\sim$ **R** (the radius of curvature of spacetime) around the worldline of the reference observer [10]. This also means that

$g'_{\mu\nu} = \eta_{\mu\nu}$ by construction. Within this region it is possible to construct the complex gravitomagntetic potentials:

$$\Phi'(T, \vec{X}) = -\frac{1}{2} R'_{0i0j} X^i X^j + ..., \tag{4.12}$$

$$A'_i(T, \vec{X}) = \frac{1}{3} R'_{0jik} X^j X^k + ..., \tag{4.13}$$

which in the real case is the expression for the four vector A_μ of gravitomagnetism (GM). The setup (4.12) allows for the following construction of the complex GM fields:

$$E'_i(T, \vec{X}) = R'_{0i0j} X^j + ..., \tag{4.14}$$

$$B'_i(T, \vec{X}) = -\frac{1}{2} \epsilon_{ijk} R'^{jk_{0l}}(T) X^l + ..., \tag{4.15}$$

which decomposes into the the general relativistic- and imaginary parts (Here E'_i and B'_i are the complex gravitoelectromagnetic fields, and G_i combined with B_{Gi} are the real gravitoelectromagnetic fields):

$$E'_i = G_i + ikE_i \tag{4.16}$$

$$B'_i = B_{Gi} + ikB_i. \tag{4.17}$$

which allows for the construction of the complex Faraday tensor (to linear order in $\vec{X}$):

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu, \tag{4.18}$$

which we can be decomposed into general relativistic and imaginary tensor components:

$$F'_{\mu\nu} = G_{\mu\nu} + ikF_{\mu\nu}. \tag{4.19}$$

Also note that $F'_{0i} = -E'_i$ and $F'_{ij} = \epsilon_{ijk} B'^k$. Then Maxwell's equations $F[\alpha\beta, \gamma] = 0$ and $F^{\alpha\beta}_{,\beta} = 4\pi J^\alpha$ are satisfied to lowest order in $|X|/\mathbf{R}$ with:

$$4\pi J'_\alpha(T, 0) = -8\pi\Big(T'_{0\alpha} - \frac{1}{2}\eta_{0\alpha} T'^{\beta}_{\beta}\Big) \tag{4.20}$$

along the trajectory in Fermi coordinates [10]. The Lorenz force for the linearized complex field appears as [12]:

$$\frac{\partial^2 X^i}{\partial T^2} + R'_{0i0j} X^j + 2R'_{ikj0} V^k X^j + \big(2R'_{0kj0} V^i V^k + \frac{2}{3} R'_{ikjl} V^k V^l + \frac{2}{3} R'_{0kjl} V^i V^k V^l\big) X^j = 0, \tag{4.21}$$

where $V^i = dX^i/dT$. For linear order velocity this becomes:

$$m\frac{\partial^2 \vec{X}}{\partial T^2} = -m\vec{E} - 2m\vec{V} \times \vec{B}, \tag{4.22}$$

where m is the mass of the test particle. This shows that the entire linearized complex field is a spin-2 field, a remnant from the spin-2 character of the complex field equations. The complex stress-energy tensor in the Fermi approach can be set up along the trajectory locally as:

$$T'^{\alpha\beta} = \frac{1}{4\pi G}\Big(F'^{\alpha}_{\gamma} F^{\beta\gamma} - \frac{1}{4} g'^{\alpha\beta} F'_{\gamma\delta} F'^{\gamma\delta}\Big) = \frac{1}{4\pi G}\Big(R'^{\alpha}_{\gamma 0i} R'^{\beta\gamma_0 j} - \frac{1}{4}\eta^{\alpha\beta} R'_{\gamma\delta 0i} R'^{\gamma\delta_{0j}}\Big) X^i X^j. \tag{4.23}$$

This stress-energy tensor is only really valid when averaged over a small domain though [10, 12]. The real parts of the stress-energy tensor (for $g_{\mu\nu} \sim \eta_{\mu\nu}$) becomes (together with the decomposition notation (4.19)):

$$Re[T'^{\alpha\beta}] = \frac{1}{4\pi G}\Big(G^{\alpha}_{\gamma} G^{\beta\gamma} - \frac{1}{4}\eta^{\alpha\beta} G_{\gamma\delta} G^{\gamma\delta} - k^2\Big(F^{\alpha}_{\gamma} F^{\beta\gamma} - \frac{1}{4}\eta^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}\Big)\Big). \tag{4.24}$$

Here the traditional gravitomagnetic part of the stress-energy tensor is visible as the first two terms. Indeed, if we let $k \to 0$ or $\alpha_{\mu\nu} \to 0$ we get:

$$T'^{\alpha\beta} = T^{\alpha\beta} = \frac{1}{4\pi G}\Big(G^{\alpha}_{\gamma}G^{\beta\gamma} - \frac{1}{4}\eta^{\alpha\beta}G_{\gamma\delta}G^{\gamma\delta}\Big). \tag{4.25}$$

which is the traditional stress-energy tensor of gravitomagnetism [10, 12].

4.3 Wave equations

As an application of the linearized equations we shall here construct the complex gravitational wave equation from the source less wave equations (from (4.12)):

$$\Box A'_{\mu} = \Box(A_{G\mu} + ikA_{E\mu}) = 0. \tag{4.26}$$

In the real valued situation equation (4.26) is a remnant from the gravitomagnetic equations that appear in the linear approach to general relativity [12]. It has the general solution:

$$A'_{\mu} = c_{\mu}e^{ik_{\nu}x^{\nu}} = c_{\mu}(\cos(k_{\nu}x^{\nu}) + i\sin(k_{\nu}x^{\nu})), \tag{4.27}$$

where c_{μ} and k_{ν} are a constant four vectors. In this formalism the complex gravity radiation is always complex.

5. Conclusions

In this paper we have investigated Moffat's symmetric complex metric generalization of general relativity. We reviewed some basic results regarding the complex symmetric field as well as its linearized retarded field equations. We showed that in the linear case the complex symmetric field equations become a complex form of gravitomagnetism (GM). As a more rigorous approach to GM we used a Fermi coordinate approach. In connection with this we derived complex gravity waves from the source less wave equations arising in the complex gravitomagnetism equations. Many open problems remain in complex symmetric gravity, like for example the extension to spin fields and torsion via Einstein-Cartan theory. Also the introduction of a non-symmetric complex field via Moyal products could perhaps lead to a quantized version of the field [14].

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References

[1] A. H. Chamseddine, Complexified gravity in non-commutative spaces, *arXiv:hep-th/0005222v2 29 May 2000*.

[2] A. H. Chamseddine, Complexified gravity in non-commutative spaces, Comm. Math. Phys., Vol 218, No.2, pp.283-292, 2001.

[3] A. H. Chamseddine, Gravity in Complex Hermitian Space-Time, *arXiv:hep-th/0610099v1 9 Oct 2006*.

[4] A. H. Chamseddine, Gravity in Complex Hermitian Space-Time, "Einstein in Alexandria, The Scientific Symposium", Biblotheca Alexandrina, pp. 39-53, 2006.

[5] A. Einstein, A Generalization of the Relativistic Theory of Gravitation, The Annals of Mathematics, Second Series, Vol. 46, No. 4, pp. 578-584, 1945.

[6] A. Einstein, A Generalization of the Relativistic Theory of Gravitation, II, The Annals of Mathematics, Second Series, Vol. 47, No. 4, pp. 731-741, 1946.

[7] H.F.M. Goenner, On the History of Unified Field Theories, Living reviews in Relativity 2004.

[8] W. F. Kao, Induced Gravity with complex metric field, *arXiv:hep-th/0008124v2 1 Aug 2001*.

[9] R. Kerr, On the Spherically Symmetric Solutions in Moffat's Unified Field Theory, Nuovo Cim. X8, 789, 1958.

[10] B. Mashhoon, Gravitoelectromagnetism, *arXiv:gr-qc/0011014v1 3 Nov 2000*.

[11] B. Mashhoon, Gravitoelectromagnetism, Reference Frames and Gravitomagnetism, Proceedings of the XXIII Spanish relativity Meeting, 2001.

[12] B. Mashhoon, Gravitoelectromagnetism: A Brief Review. *arXiv:gr-qc/0311030v2 17 Apr 2008*.

[13] B. Mashhoon, Gravitoelectromagnetism: A Brief Review, In L. Iorio, editor, Measuring Gravitomagnetism: A Challenge Enterprise, pp. 29-39. Nova Publishers, 2007.

[14] J. W. Moffat, Noncommutative Quantum Gravity, Phys. Lett. B, Vol. 491, Issues 3-4, pp. 345-352, *arXive:hep-th/0007181v2 25 Sep 2000*.

[15] J. W. Moffat, Generalized Riemann spaces, Mathematical Proceedings of the Cambridge Philosophical Society, Volume 52, Issue 04, Oct 1956, pp 623-625.

[16] J. W. Moffat, The foundations of a generalization of gravitation theory, Mathematical Proceedings of the Cambridge Philosophical Society, 53:473-488 Cambridge University Press, 1957.

[17] J. W. Moffat, The static spherically symmetric solutions in a unified field theory, Mathematical Proceedings of the Cambridge Philosophical Society, 53:489-493 Cambridge University Press, 1957.

[18] D. Pandres JR, E. L. Green, Unified field theory from enlarged transformation group. The consistent hamiltonian, *arXiv:gr-qc/0401090v1 21 Jan 2004*.

[19] D. Pandres JR, E. L. Green, Unified field theory from enlarged transformation group. The consistent hamiltonian, Int. J. of Theor. Phys., Vol 42, No.8, pp.1849-1873, 2003.

[20] B.F. Schutz, A first course in general relativity, Cambridge Univ. Press 1985.